1.

a)

$$AB = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} e & f \\ g & h \end{pmatrix} = \begin{pmatrix} ae+bg & af+bh \\ ce+dg & cf+dh \end{pmatrix}$$

$$BA = \begin{pmatrix} e & f \\ g & h \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} ae+cf & be+df \\ ag+ch & bg+dh \end{pmatrix}$$

b)

$$A(B+C) = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} e+j & f+k \\ g+l & h+m \end{pmatrix} = \begin{pmatrix} a(e+j)+b(g+l) & a(f+k)+b(h+m) \\ c(e+j)+d(g+l) & c(f+k)+d(h+m) \end{pmatrix}$$

$$AB+AC = \begin{pmatrix} ae+bg & af+bh \\ ce+dg & cf+dh \end{pmatrix} + \begin{pmatrix} aj+bl & ak+bm \\ cj+dl & ck+dm \end{pmatrix}$$

$$= \begin{pmatrix} a(e+j)+b(g+l) & a(f+k)+b(h+m) \\ c(e+j)+d(g+l) & c(f+k)+d(h+m) \end{pmatrix}$$

c)

$$(AB)C = \begin{pmatrix} ae+bg & af+bh \\ ce+dg & cf+dh \end{pmatrix}\begin{pmatrix} j & k \\ l & m \end{pmatrix} = \begin{pmatrix} j(ae+bg)+l(af+bh) & k(ae+bg)+m(af+bh) \\ j(ce+dg)+l(cf+dh) & k(ce+dg)+m(cf+dh) \end{pmatrix}$$

$$A(BC) = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} ej+fl & ek+fm \\ gj+hl & gk+hm \end{pmatrix} = \begin{pmatrix} a(ej+fl)+b(gj+hl) & a(ek+fm)+b(gk+hm) \\ c(ej+fl)+d(gj+hl) & c(ek+fm)+d(gk+hm) \end{pmatrix}$$

d)

$$AI=\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a\cdot 1+b\cdot 0 & a\cdot 0+b\cdot 1 \\ c\cdot 1+d\cdot 0 & c\cdot 0+d\cdot 1 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$$IA=\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a\cdot 1+c\cdot 0 & b\cdot 1+d\cdot 0 \\ a\cdot 0+c\cdot 1 & b\cdot 0+d\cdot 1 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

2.

$$\mathrm{Det}(A) = -4\cdot 1 - 3\cdot 3 = -13$$

$$\mathrm{Det}(B) = 1\cdot\begin{vmatrix} 5 & 6 \\ -4 & -3 \end{vmatrix} - 2\begin{vmatrix} 4 & 6 \\ -5 & -3 \end{vmatrix} + 3\begin{vmatrix} 4 & 5 \\ -5 & -4 \end{vmatrix}$$

$$= 1\cdot(-15+24) - 2(-12+30) + 3(-16+25)$$

$$= 9 - 36 + 27$$

$$= 0$$

$$\mathrm{Det}(C) = 2\cdot 3 - 0\cdot 4 = 6$$

$$\mathrm{Det}(D) = 2\cdot 3\cdot 1 = 6$$

Because D is lower triangular, the determinant is equal to the product of the diagonal elements. You can also find it the hard way:

$$2\begin{vmatrix} 3 & 8 \\ 0 & 1 \end{vmatrix} - 4\begin{vmatrix} 0 & 8 \\ 0 & 1 \end{vmatrix} + 7\begin{vmatrix} 0 & 3 \\ 0 & 0 \end{vmatrix} = 2(3\cdot 1 - 8\cdot 0) - 4(0\cdot 1 - 8\cdot 0) + 7(0\cdot 0 - 0\cdot 3) = 6$$

3. a)

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

b)

$$\begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}^{-1} = \begin{pmatrix} \frac{3}{3\cdot 1 - 0\cdot 0} & \frac{0}{3\cdot 1 - 0\cdot 0} \\ \frac{0}{3\cdot 1 - 0\cdot 0} & \frac{1}{3\cdot 1 - 0\cdot 0} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1/3 \end{pmatrix}$$

c)

$$\begin{pmatrix} 1 & 3 \\ 3 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} \frac{1}{1\cdot 1 - 3\cdot 3} & -\frac{3}{1\cdot 1 - 3\cdot 3} \\ -\frac{3}{1\cdot 1 - 3\cdot 3} & \frac{1}{1\cdot 1 - 3\cdot 3} \end{pmatrix} = \begin{pmatrix} -1/8 & 3/8 \\ 3/8 & -1/8 \end{pmatrix}$$

d)

$$\begin{pmatrix} 1 & 2 \\ 1 & 4 \end{pmatrix}^{-1} = \begin{pmatrix} \frac{4}{4-2} & -\frac{2}{4-2} \\ -\frac{1}{4-2} & \frac{1}{4-2} \end{pmatrix} = \begin{pmatrix} 2 & -1 \\ -1/2 & 1/2 \end{pmatrix}$$

e)

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}^{-1} = \begin{pmatrix} \frac{4}{4-6} & -\frac{2}{4-6} \\ -\frac{3}{4-6} & \frac{1}{4-6} \end{pmatrix} = \begin{pmatrix} -2 & 1 \\ +3/2 & -1/2 \end{pmatrix}$$

4.

$$\mathrm{Det}(M) = 1\begin{vmatrix} 4 & 3 \\ 2 & -2 \end{vmatrix} + 1\begin{vmatrix} 0 & 3 \\ 0 & -2 \end{vmatrix} + 1\begin{vmatrix} 0 & 4 \\ 0 & 2 \end{vmatrix}$$

$$= 1(-8-6) + 1(0-0) + 1(0-0)$$

$$= -14$$

$$\mathrm{Det}(D) = \begin{vmatrix} 4 & 3 \\ 2 & -2 \end{vmatrix} = -8-6 = -14$$

5.

$$M = \begin{pmatrix} -\beta-\delta & \gamma \\ \beta & -\gamma-\delta \end{pmatrix}$$

$$\vec{V} = \begin{pmatrix} -\alpha \\ 0 \end{pmatrix}$$

$$M\vec{n} = \vec{V}$$

$$\vec{n} = M^{-1}\vec{V}$$

$$\vec{n} = \begin{pmatrix} \frac{-\gamma-\delta}{(-\beta-\delta)(-\gamma-\delta)-\beta\gamma} & \frac{-\gamma}{(-\beta-\delta)(-\gamma-\delta)-\beta\gamma} \\ \frac{-\beta}{(-\beta-\delta)(-\gamma-\delta)-\beta\gamma} & \frac{-\beta-\delta}{(-\beta-\delta)(-\gamma-\delta)-\beta\gamma} \end{pmatrix}\begin{pmatrix} -\alpha \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} n_1 \\ n_2 \end{pmatrix} = \begin{pmatrix} \frac{\alpha(\gamma+\delta)}{\delta(\delta+\beta+\gamma)} \\ \frac{\alpha\beta}{\delta(\delta+\beta+\gamma)} \end{pmatrix} \Longrightarrow n_1 = \frac{\alpha(\gamma+\delta)}{\delta(\delta+\beta+\gamma)}$$

$$n_2 = \frac{\alpha\beta}{\delta(\delta+\beta+\gamma)}$$

6. a) $M = \begin{pmatrix} \alpha & 0 \\ \beta & 3\delta \end{pmatrix}$

$$M - \lambda I = \begin{pmatrix} \alpha - \lambda & 0 \\ \beta & 3\delta - \lambda \end{pmatrix}$$

$$\text{Det}(M - \lambda I) = (\alpha - \lambda)(3\delta - \lambda)$$

$$(\alpha - \lambda)(3\delta - \lambda) = 0$$

$$\Rightarrow \quad \lambda_1 = \alpha \qquad \lambda_2 = 3\delta$$

b) $\text{Det}\begin{pmatrix} 1-\lambda & 2 \\ 3 & 2-\lambda \end{pmatrix} = 0$

$$(1-\lambda)(2-\lambda) - 6 = 0$$

$$\lambda^2 - 3\lambda + 2 - 6 = 0$$

$$\lambda - 3\lambda - 4 = 0$$

$$(\lambda - 4)(\lambda + 1) = 0$$

$$\lambda_1 = 4 \quad \lambda_2 = -1$$

c) $M = \begin{pmatrix} 2 & 7 & 5 \\ 0 & 2 & 0 \\ 4 & 3 & 1 \end{pmatrix}$

If a matrix contains a row or column that is all zeros except for the element on the diagonal (m_{ii}), then one of the eigenvalues is m_{ii} and the others are the eigenvalues of the smaller matrix obtained by deleting the i^{th} row and i^{th} column of M.

So, $\lambda_1 = 2$

$$\text{Det}\begin{pmatrix} 2-\lambda & 5 \\ 4 & 1-\lambda \end{pmatrix} = 0$$

$$(2-\lambda)(1-\lambda) - 20 = 0$$

$$\lambda^2 - 3\lambda + 2 - 20 = 0$$

$$(\lambda - 6)(\lambda + 3) = 0$$

$$\lambda_2 = 6 \quad \lambda_3 = -3$$

d)

$$M = \begin{pmatrix} 1 & 2 & 0 \\ 3 & 2 & 0 \\ x & y & z \end{pmatrix}$$

Using the same rule as in (c), one of the eigenvalues is $\lambda_1 = z$. The others are given by

$$\mathrm{Det}\begin{pmatrix} 1-\lambda & 2 \\ 3 & 2-\lambda \end{pmatrix} = 0$$

$$(1-\lambda)(2-\lambda) - 6 = 0$$

$$\lambda^2 - 3\lambda + 2 - 6 = 0$$

$$\lambda^2 - 3\lambda - 4 = 0$$

$$(\lambda - 4)(\lambda + 1) = 0$$

$$\lambda_2 = 4 \qquad \lambda_3 = -1$$

e) The eigenvalues are the eigenvalues of the two submatrices

$$\begin{pmatrix} \alpha & \beta \\ \beta & \alpha \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \delta & \delta-\gamma \\ \delta-\gamma & \delta \end{pmatrix}$$

Both of these matrices are symmetric (identical above and below the diagonal), so we know all of the eigenvalues will be real.

$$\mathrm{Det}\begin{pmatrix} \alpha-\lambda & \beta \\ \beta & \alpha-\lambda \end{pmatrix} = 0$$

$$(\alpha-\lambda)^2 - \beta^2 = 0$$

$$\lambda^2 - 2\alpha\lambda + (\alpha^2+\beta^2) = 0$$

$$\lambda = \frac{2\alpha \pm \sqrt{4\alpha^2 - 4(\alpha^2+\beta^2)}}{2}$$

$$\lambda = \frac{2\alpha \pm \sqrt{4\beta^2}}{2} = \frac{2\alpha \pm 2\beta}{2} = \alpha \pm \beta$$

$$\lambda_1 = \alpha+\beta \quad , \quad \lambda_2 = \alpha-\beta$$

$$\mathrm{Det}\begin{pmatrix} \delta-\lambda & \delta-\gamma \\ \delta-\gamma & \delta-\lambda \end{pmatrix} = 0 \Rightarrow (\delta-\lambda)^2 - (\delta-\gamma)^2 = 0$$

$$\Rightarrow \lambda^2 - 2\delta\lambda + \delta^2 - \delta^2 + 2\delta\gamma - \gamma^2 = 0$$

$$\Rightarrow \lambda^2 - 2\delta\lambda + 2\delta\gamma - \gamma^2 = 0$$

$$\Rightarrow \lambda^2 - 2\delta\lambda + \gamma\lambda - \gamma\lambda + \gamma(2\delta-\gamma)$$

$$\Rightarrow (\lambda-\gamma)(\lambda-2\delta+\gamma) \Rightarrow \begin{array}{l} \lambda_3 = \gamma \\ \lambda_4 = 2\delta-\gamma \end{array}$$

7a. $\lambda_1 = \alpha$, $\lambda_2 = 3\delta$

$$\begin{pmatrix} \alpha & 0 \\ \beta & 3\delta \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \alpha\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$$

$\alpha u_1 = \alpha u_1 \longrightarrow u_1$ can be anything. Let $u_1 = 1$

$$\beta u_1 + 3\delta u_2 = \alpha u_2$$

$$\beta(1) + 3\delta u_2 = \alpha u_2$$

$$\beta = u_2(\alpha - 3\delta)$$

$$u_2 = \beta / \alpha - 3\delta$$

$$\vec{u} = \begin{pmatrix} 1 \\ \frac{\beta}{\alpha - 3\delta} \end{pmatrix}$$

$$\begin{pmatrix} \alpha & 0 \\ \beta & 3\delta \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = 3\delta\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$$

$\alpha u_1 = 3\delta u_1 \rightarrow$ Only $u_1 = 0$

$$\beta u_1 + 3\delta u_2 = 3\delta u_2$$

$0 + 3\delta u_2 = 3\delta u_2 \longrightarrow$ any u_2

So $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ is an eigenvector of $\lambda_2 = 3\delta$

and $\begin{pmatrix} 1 \\ \frac{\beta}{\alpha - 3\delta} \end{pmatrix}$ is an eigenvector of $\lambda_1 = \alpha$.

b) $\lambda_1 = 4 \qquad \lambda_2 = -1$

$$\begin{pmatrix} 1 & 2 \\ 3 & 2 \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = 4\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$$

$$u_1 + 2u_2 = 4u_1$$

$$3u_1 + 2u_2 = 4u_2$$

$$\Rightarrow 3u_1 = 2u_2$$

$$u_1 = \tfrac{2}{3}u_2$$

$$\frac{2}{3}u_2 + 2u_2 = 4\left(\frac{2}{3}u_2\right)$$

$$\frac{8}{3}u_2 = \frac{8}{3}u_2 \to \text{Any } u_2.\ \text{Let } u_2 = 1.$$

$$u_1 = \tfrac{2}{3}\cdot 1 = \tfrac{2}{3}$$

So $\begin{pmatrix} 2/3 \\ 1 \end{pmatrix}$ is an eigenvector of $\lambda_1 = 4$.

$$\begin{pmatrix} 1 & 2 \\ 3 & 2 \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = -1\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$$

$$u_1 + 2u_2 = -u_1 \Rightarrow 2u_2 = -2u_1 \Rightarrow u_2 = -u_1$$

$$3u_1 + 2u_2 = -u_2$$

$$3u_1 + 2(-u_1) = -(-u_1)$$

$$3u_1 - 2u_1 = u_1 \to \text{Any } u_1.\ \text{Let } u_1 = 1.$$

So $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ is an ~~eigenvalue~~ eigenvector of $\lambda_2 = -1$.

c) $\lambda_1 = 2 \quad \lambda_2 = 6 \quad \lambda_3 = -3$

$$\begin{pmatrix} 2 & 7 & 5 \\ 0 & 2 & 0 \\ 4 & 3 & 1 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} = 2 \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}$$

$$2u_1 + 7u_2 + 5u_3 = 2u_1$$
$$\Rightarrow u_3 = \frac{7u_2}{5}$$

$$2u_2 = 2u_2$$

$\Rightarrow u_2$ can take any value. Let $u_2 = 10$.

$$\Rightarrow u_3 = \frac{7 \cdot 10}{5} = 14$$

$$4u_1 + 3u_2 + u_3 = 2u_3$$
$$4u_1 + 3 \cdot 10 + 14 = 2 \cdot 14$$
$$4u_1 + 30 + 14 = 28$$
$$4u_1 = -16$$
$$u_1 = -4$$

So $\begin{pmatrix} -4 \\ 10 \\ 14 \end{pmatrix}$ is an eigenvector of $\lambda_1 = 2$.

$$\begin{pmatrix} 2 & 7 & 5 \\ 0 & 2 & 0 \\ 4 & 3 & 1 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} = 6 \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}$$

$$2u_1 + 7u_2 + 5u_3 = 6u_1$$

$$u_2 = 6u_2$$

$$\Rightarrow u_2 = 0$$

$$4u_1 + 3u_2 + u_3 = 6u_3$$

$$2u_1 + 7 \cdot 0 + 5u_3 = 6u_1$$

$$u_3 = \frac{4}{5} u_1$$

$$4u_1 + 3 \cdot 0 + \frac{4}{5} u_1 = 6 \cdot \frac{4}{5} u_1$$

$$\frac{20u_1}{5} + \frac{4u_1}{5} = \frac{24u_1}{5}$$

u_1 can be anything. Let $u_1 = 1$.

Then $\begin{pmatrix} 1 \\ 0 \\ 4/5 \end{pmatrix}$ is an eigenvector of $\lambda_2 = 6$.

$$\begin{pmatrix} 2 & 7 & 5 \\ 0 & 2 & 0 \\ 4 & 3 & 1 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} = -3 \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}$$

$$2u_1 + 7u_2 + 5u_3 = -3u_1$$

$$2u_2 = -3u_2$$

$$\Rightarrow u_2 = 0$$

$$4u_1 + 3u_2 + u_3 = -3u_3$$

$$4u_1 = -4u_3$$

$$u_1 = -u_3$$

$$2u_1 - 5u_1 = -3u_1$$

$$-3u_1 = -3u_1$$

$\Rightarrow u_1$ can be anything

So $\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$ is an eigenvector of $\lambda_3 = -3$.

8. a) $M = \begin{pmatrix} 4 & 1 \\ 3 & 2 \end{pmatrix}$

$$\text{Det} \begin{pmatrix} 4-\lambda & 1 \\ 3 & 2-\lambda \end{pmatrix} = 0$$

$$(4-\lambda)(2-\lambda) - 3 = 0$$

$$\lambda^2 - 6\lambda + 8 - 3 = 0$$

$$(\lambda - 5)(\lambda - 1) = 0$$

$$\lambda_1 = 5 \quad \lambda_2 = 1$$

$$\begin{pmatrix} 4 & 1 \\ 3 & 2 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = 5 \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$$

$$4u_1 + u_2 = 5u_1$$

$$\Rightarrow u_2 = u_1$$

$$3u_1 + 2u_2 = 5u_2$$

$$5u_2 = 5u_2$$

So $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ is an right eigenvector for $\lambda_1 = 5$

$$\begin{pmatrix} 4 & 1 \\ 3 & 2 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$$

$$4u_1 + u_2 = u_1$$

$$3u_1 + 2u_2 = u_2$$

$$\Rightarrow u_2 = -3u_1$$

$$4u_1 - 3u_1 = u_1$$

$$u_1 = u_1$$

So $\begin{pmatrix} 1 \\ -3 \end{pmatrix}$ is an right eigenvector for $\lambda_2 = 1$

$$(v_1 \quad v_2)\begin{pmatrix} 4 & 1 \\ 3 & 2 \end{pmatrix} = (v_1 \quad v_2)\, 5$$

$$4v_1 + 3v_2 = 5v_1$$

$$v_1 + 2v_2 = 5v_2$$

$$\Rightarrow v_1 = 3v_2$$

$$4v_1 + v_1 = 5v_1$$

$$5v_1 = 5v_1$$

$$\Rightarrow v_1 = 1$$

So $(1 \quad 1/3)$ is a left eigenvector of $\lambda_1 = 5$

$$(v_1 \quad v_2)\begin{pmatrix} 4 & 1 \\ 3 & 2 \end{pmatrix} = (v_1 \quad v_2)$$

$$4v_1 + 3v_2 = v_1$$

$$v_1 + 2v_2 = v_2$$

$$\Rightarrow v_1 = -v_2$$

$$4v_1 - 3v_1 = v_1$$

$$\Rightarrow v_1 = 1$$

So $(1 \quad -1)$ is a left eigenvector of $\lambda_2 = 1$.

Thus $\lambda_1 = 5$, $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 1/3 \end{pmatrix}$ and

$\lambda_2 = 1$, $\begin{pmatrix} 1 \\ -3 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ are the solutions.

b) $M = \begin{pmatrix} -3 & 6 \\ 0 & 2 \end{pmatrix}$

Because M is upper triangular, the eigenvalues are -3 and 2.

Right eigenvectors:

$$\begin{pmatrix} -3 & 6 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = -3\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$$

$$-3u_1 + 6u_2 = -3u_1$$

$$u_2 = -3u_2$$

$$\Rightarrow u_2 = 0$$

$$-3u_1 = -3u_1$$

$\Rightarrow u_1 =$ anything

So $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ is a right eigenvector of $\lambda_1 = -3$

$$\begin{pmatrix} -3 & 6 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = 2\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$$

$$-3u_1 + 6u_2 = 2u_1$$

$$2u_2 = 2u_2$$

$\Rightarrow u_2 =$ anything – let $u_2 = 5$.

$$-3u_1 + 30 = 2u_1$$

$$\Rightarrow u_1 = 6$$

So $\begin{pmatrix} 6 \\ 5 \end{pmatrix}$ is a right eigenvector of $\lambda_2 = 2$.

Left eigenvectors:

$$\begin{pmatrix} v_1 & v_2 \end{pmatrix} \begin{pmatrix} -3 & 6 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} v_1 & v_2 \end{pmatrix} -3$$

$$-3v_1 = -3v_1$$

$\Rightarrow v_1 =$ anything. Let $v_1 = 5$.

$$6v_1 + 2v_2 = -3v_2$$

$$30 + 2v_2 = -3v_2$$

$$v_2 = 6$$

So $\begin{pmatrix} 5 \\ 6 \end{pmatrix}$ is a left eigenvector for $\lambda_1 = -3$.

$$\begin{pmatrix} v_1 & v_2 \end{pmatrix} \begin{pmatrix} -3 & 6 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} v_1 & v_2 \end{pmatrix} 2$$

$$-3v_1 = 2v_1$$

$$\Rightarrow v_1 = 0$$

$$6v_1 + 2v_2 = 2v_2$$

$$2v_2 = 2v_2$$

$\Rightarrow v_2 =$ anything. Let $v_2 = 1$.

Then $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ is a left eigenvector for $\lambda_2 = 2$.