Lotka-Volterra model

$$\dot{N} = rN - aNP$$

$$\dot{P} = baNP - mP$$

Equilibria : $(\hat{N}=0, \hat{P}=0)$ is an equilibrium

$$\dot{N} = 0 \Rightarrow r\hat{N} - a\hat{N}\hat{P} = 0$$

$$\hat{P} = \frac{r}{a}$$

$$\dot{P} = 0 \Rightarrow ba\hat{N}\hat{P} - m\hat{P} = 0$$

$$\hat{N} = \frac{m}{ba}$$

$\left(\hat{N} = \frac{m}{ba}, \hat{P} = \frac{r}{a}\right)$ is an equilibrium

Stability is given by the Jacobian

$$J = \begin{pmatrix} r - aP & -aN \\ baP & baN - m \end{pmatrix}$$

Evaluated at $(0,0)$:

$$J\Big|_{(0,0)} = \begin{pmatrix} r & 0 \\ 0 & -m \end{pmatrix}.$$

The eigenvalues of this matrix are $r, -m$. Since $r>0$, the equilibrium is unstable.

Evaluated at $\left(\frac{m}{ba}, \frac{r}{a}\right)$.

$$J\Big|_{\left(\frac{m}{ba}, \frac{r}{a}\right)} = \begin{pmatrix} 0 & -\frac{m}{b} \\ br & 0 \end{pmatrix}.$$

The eigenvalues of this matrix are given by

$$\lambda_{1,2} = \frac{Tr\,J \pm \sqrt{Tr\,J^2 - 4\,Det\,J}}{2}$$

$$\lambda_{1,2} = \frac{0 \pm \sqrt{-4mr}}{2}$$

Thus, the two eigenvalues have 0 real part, implying that the equilibrium is <u>neutrally stable</u>.

The other predator-prey system is

$$\dot{N} = rN\left(1 - \frac{N}{K}\right) - aNP$$

$$\dot{P} = baNP - mP$$

<u>Equilibria</u>

$(\hat{N} = 0, \hat{P} = 0)$ is an equilibrium.

However, $(\hat{N} = K, \hat{P} = 0)$ is also an equilibrium. If the predator goes extinct, the prey population will grow to its carrying capacity.

$$\dot{P} = 0 \implies ba\hat{N}\hat{P} - m\hat{P} = 0$$

$$\hat{N} = \frac{m}{ba}$$

$$\dot{N} = 0 \implies r\hat{N}\left(1 - \frac{\hat{N}}{K}\right) - a\hat{N}\hat{P} = 0$$

$$\hat{P} = \frac{r}{a}\left(1 - \frac{\hat{N}}{K}\right)$$

$$\hat{P} = \frac{r}{a}\left(1 - \frac{m}{baK}\right)$$

$$\hat{P} = \frac{r}{a}\left(\frac{baK - m}{baK}\right)$$

Notice that a biologically feasible $\hat{P}$ requires that $baK - m > 0$ (else $\hat{P} < 0$, which is biologically meaningless).

Stability

The Jacobian is

$$J = \begin{pmatrix} r\left(1 - \frac{N}{K}\right) - \frac{rN}{K} - aP & -aN \\ baP & baN - m \end{pmatrix}$$

At $(0,0)$

$$J\Big|_{(0,0)} = \begin{pmatrix} r & 0 \\ 0 & -m \end{pmatrix}$$

Since the eigenvalues are $\lambda_1 = r$, $\lambda_2 = -m$, this equilibrium is unstable for $r > 0$.

At $(K,0)$

$$J\Big|_{(K,0)} = \begin{pmatrix} -r & -aK \\ 0 & baK - m \end{pmatrix}.$$

The eigenvalues are $\lambda_1 = -r$, $\lambda_2 = baK - m$, so the equilibrium will be <u>unstable</u> if $baK - m > 0$ (the condition for $\hat{P} > 0$). $baK - m$ is the predator's per-capita population growth rate when prey abundance is at its maximum. Thus, $baK - m$ is the fastest the predator population could ever grow. If this is negative, the predator will have a negative growth rate for all prey population sizes!

$$J\Big|_{\hat{N}, \hat{P}} = \begin{pmatrix} r\left(1 - \frac{m}{baK}\right) - \frac{rm}{baK} - a\left(\frac{r}{a}\left(\frac{baK-m}{baK}\right)\right) & -\frac{m}{b} \\ ba\left(\frac{r}{a}\left(\frac{baK-m}{baK}\right)\right) & 0 \end{pmatrix}$$

$$= \begin{pmatrix} r - \frac{2rm}{baK} - r + \frac{rm}{baK} & -\frac{m}{b} \\ \frac{r}{a}\left(\frac{baK-m}{K}\right) & 0 \end{pmatrix}$$

$$= \begin{pmatrix} -\frac{rm}{baK} & -\frac{m}{b} \\ \frac{r}{aK}(baK-m) & 0 \end{pmatrix}$$

$$\mathrm{Tr}\, J = -\frac{rm}{baK}$$

$$\mathrm{Det}\, J = \frac{rm}{baK}(baK - m)$$

Since stability requires $\mathrm{Tr}\, J < 0$ and $\mathrm{Det}\, J > 0$, this equilibrium will be stable as long as $bak - m > 0$.

$$\frac{dS}{dt} = \mu - \beta S I - \mu S + \gamma R$$

$$\frac{dI}{dt} = \beta S I - \mu I - \rho I$$

$$\frac{dR}{dt} = \rho I - \gamma R - \mu R$$

μ is both the birth & death rate, so total population size is constant in this model.

The model is appropriate for seasonal flu because the disease does not cause any additional mortality. The disease also confers immunity (the R class), but this immunity is not permanent - individuals lose their immunity and return to the susceptible class. This is much like seasonal flu, where getting the flu one winter will typically protect you from getting the flu again that winter, but will not protect you the following winter.

<u>Equilibria of the model</u>

Because the model's state variables (S, I, R) are fractions of the total population, $(0,0,0)$ is not an equilibrium of the model.

However, if there is no disease, eventually everyone in the population will be in the susceptible class, so $(\hat{S}=1,\ \hat{I}=0,\ \hat{R}=0)$ is an equilibrium.

The endemic equilibrium with all 3 present is

$$\cancel{S}\ \dot{I}=0 \Rightarrow \beta\hat{S}\hat{I}-\mu\hat{I}-\rho\hat{I}=0$$

$$\boxed{\hat{S}=\frac{\mu+\rho}{\beta}}$$

$$\dot{R}=0 \Rightarrow \rho\hat{I}-\gamma\hat{R}-\mu\hat{R}=0$$

$$\boxed{\hat{I}=\frac{(\gamma+\mu)\hat{R}}{\rho}}$$

$$\dot{S}=0 \Rightarrow \mu-\beta\hat{S}\hat{I}-\mu\hat{S}+\gamma\hat{R}=0$$

$$\mu-\beta\left(\frac{\mu+\rho}{\beta}\right)\left(\frac{(\gamma+\mu)\hat{R}}{\rho}\right)-\frac{\mu(\gamma+\mu)\hat{R}}{\rho}-(\gamma+\mu)\hat{R}=0$$

$$\mu=\frac{(\mu+\rho)(\gamma+\mu)}{\rho}\hat{R}+\frac{\mu(\gamma+\mu)}{\rho}\hat{R}+(\gamma+\mu)\hat{R}$$

$$\boxed{\hat{R}=\frac{\mu}{\frac{(\mu+\rho)(\gamma+\mu)}{\rho}+\frac{\mu(\gamma+\mu)}{\rho}+(\gamma+\mu)}}$$

The Jacobian is

$$J = \begin{pmatrix} -\beta I - \mu & -\beta S & \gamma \\ \beta I & \beta S - \mu - \rho & 0 \\ 0 & \rho & -\gamma - \mu \end{pmatrix}$$

At $(\hat{S} = 1, \hat{I} = 0, \hat{R} = 0)$

$$J = \begin{pmatrix} -\mu & -\beta & \gamma \\ 0 & \beta - \mu - \rho & 0 \\ 0 & \rho & -\gamma - \mu \end{pmatrix}$$

Because J has a row with 0 on every entry except the diagonal, the diagonal entry $(\beta - \mu - \rho)$ is an eigenvalue. The other two eigenvalues are those of the matrix

$$\begin{pmatrix} -\mu & \gamma \\ 0 & -\gamma - \mu \end{pmatrix}$$

Thus the three eigenvalues are

$$\lambda_1 = \beta - \mu - \rho$$

$$\lambda_2 = -\mu$$

$$\lambda_3 = -\gamma - \mu.$$

Since λ_2 and λ_3 are both negative, stability depends only on λ_1. The equilibrium will be stable if

$$\beta - \mu - \rho < 0$$

or equivalently, if

$$\frac{\beta}{\mu - \rho} < 1 .$$

But $\frac{\beta}{\mu - \rho}$ is the number of new infections produced by each infected person when the population is fully susceptible.